

## An Improved Estimate of the “Attractive Region” and Modelling of Fluctuation Spectra

H. Tasso

Max-Planck-Institut für Plasmaphysik,  
EURATOM Association, Garching

Z. Naturforsch. **44a**, 325–326 (1989);  
received March 3, 1989

A previous estimate made by the author for the attractive region of a system of van der Pol-like oscillators is now improved. It leads to a richer class of attracting systems whose statistics generates a larger class of fluctuation spectra.

The system of van der Pol-like oscillators introduced by the author [1] was discussed in [2], particularly with respect to its ability to model noise and possibly turbulence. One of the conclusions of [2] is that the bounds of the attractive region derived in [1] should be improved in order to allow one to model fluctuation spectra other than equipartition of the amplitudes of the oscillators. In this note it is shown how to make a first improvement of the estimate of the attractive region.

The system introduced in [1] is

$$\ddot{Y} + [(Y, AY)M + (\dot{Y}, B\dot{Y})N - P]\dot{Y} + CY = 0. \quad (1)$$

The vector  $Y$  and the matrices,  $A, B, C, M, N, P$  were all defined in [1], as also are their largest and lowest eigenvalues, so that it is not necessary to repeat that here. Inequalities (3) and (4) of [1], which allow Lyapunov stability to be proved, are modified in the following way:

$$\frac{1}{2} \frac{d}{dt} [(\dot{Y}, \dot{Y}) + (Y, CY)] \leq -\beta_0 v_0 \left[ (\dot{Y}, \dot{Y}) + (Y, CY) + \left( Y, \left( \frac{\mu_0}{\beta_0 v_0} A - C \right) Y \right) - \frac{\pi_1}{\beta_0 v_0} \right] (\dot{Y}, \dot{Y}), \quad (2)$$

$$\frac{1}{2} \frac{d}{dt} [(\dot{Y}, \dot{Y}) + (Y, CY)] \geq -\beta_1 v_1 \left[ (\dot{Y}, \dot{Y}) + (Y, CY) + \left( Y, \left( \frac{\mu_1}{\beta_1 v_1} A - C \right) Y \right) - \frac{\pi_0}{\beta_1 v_1} \right] (\dot{Y}, \dot{Y}). \quad (3)$$

From inequality (3) we have instability around the origin, and in the case

$$\left( \frac{\mu_1}{\beta_1 v_1} A_s - C \right) \leq 0 \quad (4)$$

the instability persists if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \leq \frac{\pi_0}{\beta_1 v_1}, \quad (5)$$

$A_s$  being the symmetric part of  $A$ .

From inequality (2) and

$$\left( \frac{\mu_0}{\beta_0 v_0} A_s - C \right) \geq 0 \quad (6)$$

it can be seen that the system is stable if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \geq \frac{\pi_1}{\beta_0 v_0}. \quad (7)$$

Conditions (4) and (6) are less severe than the corresponding conditions (5) and (7) of [1]. This can easily be seen by squeezing the attractive region down to zero, i.e.

$$\frac{\pi_0}{\beta_1 v_1} = \frac{\pi_1}{\beta_0 v_0}. \quad (8)$$

From (8) it follows that

$$\pi_0 = \pi_1 \quad \text{and} \quad \beta_1 v_1 = \beta_0 v_0 = \beta v. \quad (9)$$

Since  $A_s, C, \beta v, \mu_0$  and  $\mu_1$  are assumed to be positive (see [1]), it follows from (4), (6) and (9) that

$$\mu_0 = \mu_1 = \mu \quad \text{and} \quad C = \frac{\mu}{\beta v} A_s. \quad (10)$$

Condition (10) is less restrictive than the corresponding relation (12) in [1] also obtained for a squeezed attractive region. The difference is that  $C$  and  $A_s$  do not need to be proportional to the identity. If  $A_s$  is taken as

$$A_s = \sigma I,$$

then it follows that

$$C = \frac{\alpha \mu}{\beta v} I, \quad \text{as in [1]}. \quad (11)$$

The generalization (10) has a rather important impact. First, it allows more general global attractors to be

Reprint requests to Dr. H. Tasso, Max-Planck-Institut für Plasmaphysik – Bibliothek, D-8046 Garching b. München.

0932-0784 / 89 / 0400-0325 \$ 01.30/0. – Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

considered for which a Liouville theorem such as in [3] can be proved. Second, the “shell” relation (see [2]), which is given by

$$(\dot{Y}, \dot{Y}) + (Y, CY) = \text{const}, \quad (12)$$

contains a general  $C$ . The Gibbs statistics constructed on this basis leads to

$$\langle y_i^2 \rangle = 1/\lambda_i, \quad (13)$$

where the  $\langle y_i^2 \rangle$  are the expectations of the amplitudes of the oscillators, and  $\lambda_i$  the eigenvalues of  $C$ . This is to be compared with equipartition or  $\langle y_i^2 \rangle = \text{const.}$  of [2], in which  $C$  is proportional to the identity.

If the attracting system is linear, the eigenvalues  $\lambda_i$  can be interpreted as eigenfrequencies  $f_i$  squared, or  $\lambda_i = f_i^2$ . If, however, nonlinear terms are kept in the

gyroscopic part of the attracting system as in [2] and [3], then one has  $\lambda_i \neq f_i^2$ . In that case the frequency spectrum could be obtained by evaluating the time correlation function for the nonlinear attracting system.

Finally, if other nonlinear terms similar to those mentioned at the end of [1] are added to system (1), a richer class of spectra can be generated. Also the search for other Lyapunov functions for system (1) could lead to an even better estimate of the attractive region and presumably to an even larger class of fluctuation spectra. Whether such classes are rich enough to model noise and turbulence is open, though magnetic fluctuations spectra like  $1/f^2$  are reported in the literature [4].

[1] H. Tasso, Z. Naturforsch. **41a**, 987 (1986).

[2] H. Tasso, to be published in: Proc. of IIIrd International Workshop on Mathematical Aspects of Fluids and Plasma Dynamics (Salice Terme 26–30 Sept. 1988). See also IPP 6/277 (Oct. 1988).

[3] H. Tasso, Z. Naturforsch. **42a**, 1377 (1987).

[4] P. A. Duperrex et al. Phys. Lett. **106A**, 133 (1984).