An Improved Estimate of the "Attractive Region" and Modelling of Fluctuation Spectra

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A previous estimate made by the author for the attractive region of a system of van der Pol-like oscillators is now improved. It leads to a richer class of attracting systems whose statistics generates a larger class of fluctuation spectra.

The system of van der Pol-like oscillators introduced by the author [1] was discussed in [2], particularly with respect to its ability to model noise and possibly turbulence. One of the conclusions of [2] is that the bounds of the attractive region derived in [1] should be improved in order to allow one to model fluctuation spectra other than equipartition of the amplitudes of the oscillators. In this note it is shown how to make a first improvement of the estimate of the attractive region.

The system introduced in [1] is

$$\ddot{Y} + [(Y, AY)\, M + (\dot{Y}, B\dot{Y})\, N - P]\, \dot{Y} + CY = 0\,. \tag{1}$$

The vector Y and the matrices, A, B, C, M, N, P were all defined in [1], as also are their largest and lowest eigenvalues, so that it is not necessary to repeat that here. Inequalities (3) and (4) of [1], which allow Lyapunov stability to be proved, are modified in the following way:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\dot{Y}, \dot{Y}) + (Y, CY)] \leqq -\beta_0 \nu_0 \left[(\dot{Y}, \dot{Y}) + (Y, CY) + \left(Y, \left(\frac{\mu_0}{\beta_0 \nu_0} A - C\right) Y\right) - \frac{\pi_1}{\beta_0 \nu_0}\right] (\dot{Y}, \dot{Y})\,, \tag{2}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\dot{Y}, \dot{Y}) + (Y, CY)] \geqq -\beta_1 \nu_1 \left[(\dot{Y}, \dot{Y}) + (Y, CY) + \left(Y, \left(\frac{\mu_1}{\beta_1 \nu_1} A - C\right) Y\right) - \frac{\pi_0}{\beta_1 \nu_1}\right] (\dot{Y}, \dot{Y})\,. \tag{3}$$

From inequality (3) we have instability around the origin, and in the case

$$\left(\frac{\mu_1}{\beta_1 \nu_1} A_s - C\right) \leqq 0 \tag{4}$$

the instability persists if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \leqq \frac{\pi_0}{\beta_1 \nu_1}\,, \tag{5}$$

A_s being the symmetric part of A.

From inequality (2) and

$$\left(\frac{\mu_0}{\beta_0 \nu_0} A_s - C\right) \geqq 0 \tag{6}$$

it can be seen that the system is stable if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \geqq \frac{\pi_1}{\beta_0 \nu_0}\,. \tag{7}$$

Conditions (4) and (6) are less severe than the corresponding conditions (5) and (7) of [1]. This can easily be seen by squeezing the attractive region down to zero, i.e.

$$\frac{\pi_0}{\beta_1 \nu_1} = \frac{\pi_1}{\beta_0 \nu_0}\,. \tag{8}$$

From (8) it follows that

$$\pi_0 = \pi_1 \quad \text{and} \quad \beta_1 \nu_1 = \beta_0 \nu_0 = \beta \nu\,. \tag{9}$$

Since A_s, C, $\beta \nu$, μ_0 and μ_1 are assumed to be positive (see [1]), it follows from (4), (6) and (9) that

$$\mu_0 = \mu_1 = \mu \quad \text{and} \quad C \frac{\mu}{\beta \nu} A_s\,. \tag{10}$$

Condition (10) is less restrictive than the corresponding relation (12) in [1] also obtained for a squeezed attractive region. The difference is that C and A_s do not need to be proportional to the identity. If A_s is taken as

$$A_s = \sigma I\,,$$

then it follows that

$$C = \frac{\alpha \mu}{\beta \nu} I\,, \quad \text{as in [1].} \tag{11}$$

The generalization (10) has a rather important impact. First, it allows more general global attractors to be

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considered for which a Liouville theorem such as in [3] can be proved. Second, the "shell" relation (see [2]), which is given by

$$(\dot{Y}, \dot{Y}) + (Y, CY) = \text{const}\,, \tag{12}$$

contains a general C. The Gibbs statistics constructed on this basis leads to

$$\langle y_i^2 \rangle = 1/\lambda_i\,, \tag{13}$$

where the $\langle y_i^2 \rangle$ are the expectations of the amplitudes of the oscillators, and λ_i the eigenvalues of C. This is to be compared with equipartition or $\langle y_i^2 \rangle = \text{const.}$ of [2], in which C is proportional to the identity.

If the attracting system is linear, the eigenvalues λ_i can be interpreted as eigenfrequencies f_i squared, or $\lambda_i = f_i^2$. If, however, nonlinear terms are kept in the gyroscopic part of the attracting system as in [2] and [3], then one has $\lambda_i \neq f_i^2$. In that case the frequency spectrum could be obtained by evaluating the time correlation function for the nonlinear attracting system.

Finally, if other nonlinear terms similar to those mentioned at the end of [1] are added to system (1), a richer class of spectra can be generated. Also the search for other Lyapunov functions for system (1) could lead to an even better estimate of the attractive region and presumably to an even larger class of fluctuation spectra. Whether such classes are rich enough to model noise and turbulence is open, though magnetic fluctuations spectra like $1/f^2$ are reported in the literature [4].

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